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## Skyrmion solutions by generalization of the Atiyah-Manton ansatz

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### Abstract

We generalize the approach of Atiyah and Manton for generating Skyrmion configurations from instantons. The result is a series whose parameters are found directly from the chiral angle equation. The series converge rapidly to the exact solution for a class of the Skyrme-like models (including the Skyrme model itself) but describe with less accuracy other types of models. We describe the procedure and discuss its advantages and limitations.

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## I. INTRODUCTION

As a non-linear theory of pions, the Skyrme model [1] provides an approximate description of hadronic physics in the low-energy limit. In this theory, the nucleon emerges as a bound state of the field, or more precisely as a soliton. The original Skyrme Lagrangian was of fourth order in field derivatives and it reached a 30% accuracy with respect to physical observables. Since then, we proposed along with other authors [2–5] a number of alternative models which preserved the form of the original Lagrangian while extending it to higher orders. The result was to incorporate effects due to higher-spin mesons and thus to improve the fit of most observables.

All those models have one handicap in common however, they do not admit exact analytic solutions. Very few soliton solutions are known as a matter of fact, with the exclusion of the one-dimensional sine-Gordon equation, the KdV equation and some other special cases. This is a natural consequence of non-linearity, a necessary constraint for a soliton-like solution to exist.

In the absence of analytical solutions, the only alternative to numerical treatment is the use of aptly-chosen analytical forms which provide a sometimes sufficient approximation. Apart from greatly simplifying calculation of physical quantities, a great deal of information can be extracted from such an analytic form. For instance, symmetries and general behavior of the solution are much easier to analysis, and the individual features proper to each Skyrme-like model also become more explicit. Then, an analytic form proves useful in the stability analysis of the soliton, both classical and quantum, and in the calculation of multi-Skyrmion interactions. In particular, Igarashi et al. [6] have analyzed the quantum behavior of the Skyrme model soliton on the basis of a family of trial functions, taking account of breathing motions and spin-isospin rotations whereas Hosaka et al. [7] have examined to the two Skyrmion interactions.

The exact form of most trial functions is largely arbitrary however, with the sole constraint that they have the same asymptotic behavior as the numerical solution (i.e.,  $F(r) = N\pi - \alpha r + \mathcal{O}(r^3)$  when  $r \rightarrow 0$  and  $F(r) = \beta r^{-2} + \mathcal{O}(r^{-4})$  when  $r \rightarrow \infty$ ) and that they reproduce the numerical results with a degree of accuracy. A few years ago on the other hand, Atiyah and Manton brought forward the idea of generating Skyrmion solutions from Yang-Mills instantons [8]. Using this approach as a starting point, we have developed a systematic procedure in order to parameterize any Skyrmion solution.

We first present the Atiyah-Manton idea and discuss its successes and drawbacks. We then introduce our own systematic procedure for parameterizing the solution and apply it directly to a few Skyrme-like models in the  $N = 1$  case of a single Skyrmion configuration (Section 3), and in the more general case of  $N > 1$  (Section 4). We finally offer some comments on the advantages and limitations of our method and on various ways of improving it (Section 5) for models where the chiral angle shows non-smooth behavior due to its nonlinear nature, .

## II. THE ATIYAH-MANTON IDEA

The Atiyah-Manton idea is based on topological similarities between Skyrmions and instantons. Before introducing it, let us first review the Skyrme model itself. The pion fields

are represented by a unitary  $SU(2)$  matrix denoted  $U(\mathbf{x})$ . In its familiar hedgehog form, it is expressed as follows:

$$U(\mathbf{x}) = \exp [i\tau \cdot \hat{\mathbf{x}}F(r)] \quad (1)$$

where  $F(r)$  is called the chiral angle or profile function of the solution. This field configuration constitutes a map from physical space  $R^3$  into the group manifold  $SU(2)$  and is assumed to go to the trivial vacuum for asymptotically large distances. We therefore impose  $U(r \rightarrow \infty) \rightarrow 1$ . From this last condition, one may derive the existence of a topological invariant associated with the mapping. The originality of Skyrme's idea was to identify this invariant, i.e. the winding number, with the baryon number.

Introducing the notation  $L_\mu = U^\dagger \partial_\mu U$ , the Skyrme Lagrangian for zero pion mass takes the form:

$$\mathcal{L} = -\frac{F_\pi^2}{4} \text{Tr} L_\mu L^\mu + \frac{1}{32e^2} \text{Tr} [L_\mu, L_\nu]^2 \quad (2)$$

where the first term coincides with the non-linear sigma model and the second one acts as a stabilizer.

The Yang-Mills instanton follows from a different path. We let  $A_\mu(x)$  be a gauge field and  $x = (\mathbf{x}, t)$  be a vector in  $R^4$ . We may then introduce the unitary  $SU(2)$  field  $U(x)$  in terms of the temporal component of  $A_\mu(x)$  by the following equation:

$$U^{-1}(x) \partial_t U(x) = -A_t(x)$$

This equation is solved through the method of time-ordered products by:

$$U(x) = \pm \mathcal{T} \exp \left[ - \int_{-\infty}^t A_t(x) dt \right] \quad (3)$$

with  $U(\mathbf{x}, -\infty) = -1$  and  $U(\mathbf{x}, +\infty) = U(\mathbf{x})$ .

The  $A_t$  component of the Yang-Mills field may furthermore be expressed in terms of the scalar superpotential  $\rho$ :

$$A_t = \frac{i}{2} \tau \cdot \frac{\nabla \rho}{\rho} \quad (4)$$

The so-called 't Hooft instantons of topological number  $k$  are then obtained by setting:

$$\rho = 1 + \sum_{i=1}^k \frac{\lambda_i^2}{(x - X_i)^2} \quad (5)$$

where the  $X_i$  are poles in  $R^4$  and the  $\lambda_i$  are positive constants acting as scales.

The Atiyah-Manton idea then follows from identifying the field  $U(\mathbf{x}) = U(\mathbf{x}, +\infty)$  with the Skyrme field, with  $B=k$ . In the  $B=1$  case of a single soliton centered at the origin ( $X_i = 0$ ), we set:

$$\rho = 1 + \frac{\lambda^2}{r^2 + t^2} \quad (6)$$

and find by (3) and (4):

$$F(r) = \pi \left( 1 - \frac{r}{\sqrt{r^2 + \lambda^2}} \right) \quad (7)$$

This function has the same asymptotic behavior as the numerical Skyrme solution:

$$F(r) = \begin{cases} \pi + \alpha r + O(r^3), & \text{when } r \rightarrow 0 \\ \beta r^{-2} + O(r^{-4}), & \text{when } r \rightarrow \infty \end{cases}$$

with  $\alpha = -\pi/\lambda$  and  $\beta = \pi\lambda^2/2$ . The  $\lambda$  parameter, which gives a measure of the size of the soliton, is determined by minimizing the static mass  $M_S$  of the soliton. Applying this function to the Skyrme model, we find after minimization  $M_S = 104.1 (F_\pi/e)$ , only 0.9% higher than the exact numerical result.

Despite this relatively impressive success however, this approach leaves something to be desired. For one thing, the slopes at  $r \rightarrow 0$  and  $r \rightarrow \infty$  both depend on the same parameter  $\lambda$ , which means that their values cannot be adjusted independently. More precisely, we have:  $\beta = \frac{\pi^3}{2\alpha}$ . Secondly, we find that the predictive power of this simple function rapidly decreases when we consider more elaborate Skyrme-like models such as those we present below.

Aiming to improve the fit of the exact solution while preserving the general form of the Atiyah-Manton ansatz, we consider an extension of the series. The following form turns out to simplify the calculations:

$$F(r, \lambda) = \pi \left[ 1 - \frac{r}{\sqrt{r^2 + \lambda^2}} \sum_{n=0}^{\infty} c_{2n+1} \frac{\lambda^{2[\frac{n+1}{2}]} r^{2[\frac{n}{2}]}}{(r^2 + \lambda^2)^n} \right] \quad (8)$$

where  $[z]$  is the integer part of  $z$  and  $c_0 = 1$ . This solution correspond to a general potential of the form:

$$\rho = \exp \left[ -\ln Q + \sum_{m=1}^{\infty} d_m Q^m \right] \quad \text{where, } Q = \frac{r^2 + t^2}{r^2 + t^2 + \lambda^2}.$$

where for  $d_m = 0$ , one recovers the expression in (6).

This new function is still characterized by a scale  $\lambda$ , as well as by the coefficients  $c_n$ , acting as weights with respect to the corrective terms. As we will show those coefficients are not completely free; their value is determined once a choice has been made for the slopes  $\alpha$  and  $\beta$ . The scale  $\lambda$  is once again determined by minimization of the mass of the soliton.

### III. THE N=1 SOLITONS

It proves useful to express the Lagrangian of all Skyrme-like models in the form (for the hedgehog solution):

$$\mathcal{L} = \sum_{m=1}^{\infty} h_m a^{m-1} [3a + m(b-a)] \quad (9)$$

where  $a = r^{-2} \sin^2 F$ ,  $b = F'^2$  and where we have made the scale change:  $r \rightarrow \frac{eF_\pi}{\sqrt{2}} r$ . The static energy is then written as:

$$M_S = 4\pi\left(\frac{F_\pi}{e}\right) \int_0^\infty r^2 dr \sum_{m=1}^\infty h_m a^{m-1} [3a + m(b-a)]$$

or as:

$$M_S = 4\pi\left(\frac{F_\pi}{e}\right) \int_0^\infty r^2 dr [3\chi(a) + (b-a)\chi'(a)] \quad (10)$$

where  $\chi(x) = \sum_{m=1}^\infty h_m x^m$  and  $\chi'(x) = \frac{d\chi}{dx}$ . Using the same notation, the chiral equation becomes:

$$0 = \chi'(a)[F'' + 2\frac{F'}{r} - 2\frac{\sin F \cos F}{r^2}] + a\chi''(a)[-2\frac{F'}{r} + F'^2\frac{\cos F}{\sin F} + \frac{\sin F \cos F}{r^2}]. \quad (11)$$

The Skyrme Lagrangian corresponds to the case  $\chi(a) = \chi_S(a) = a + \frac{1}{2}a^2$ .

More general models should include higher order terms [3]. One could consider the model for exponential or truncated geometric series:

$$\begin{aligned} \chi_1(a) &= e^a - 1 = a + \frac{1}{2}a^2 + \frac{1}{3!}a^3 + \dots \\ \chi_{2,M}(a) &= a - \frac{1}{2}a^2 + \frac{1}{3}a^3 - \dots - \frac{1}{M}a^M \end{aligned} \quad (12)$$

which corresponds to the choice  $h_{m \leq M} = \frac{(-)^{m-1}}{m}$ .

Another interesting model has been proposed by Gustaffson and Riska [5] which replaces the fourth order Skyrme term by a higher order term of order  $m$ :

$$\chi_{3,M}(a) = a + \frac{1}{M}a^M$$

Some other alternatives are due to Jackson et al [4]. They are:

$$\begin{aligned} \chi_4(a) &= \ln(1+a) + \frac{1}{2}a^2 \\ \chi_5(a) &= \frac{1}{4}[1 - e^{-2a}] + \frac{1}{2}a + \frac{1}{2}a^2 \\ \chi_6(a) &= a + \frac{a^3}{3+2a} \\ \chi_7(a) &= a + \frac{a^3}{3+4a} + \frac{a^4}{1+4a^2} \end{aligned}$$

In this work, we shall limit ourselves to these seven models, taking  $M=20$  for the  $\chi_{2,M}$  model and  $M=20$  for the  $\chi_{3,M}$  model.

In order to fix the coefficients in the parameterization, we use the chiral angle equation in the  $r \rightarrow 0$  and  $r \rightarrow \infty$  limits. The first step of our procedure is to express the chiral angle  $F(r)$  as a Taylor expansion, valid for small  $r$ :

$$F(r) = N\pi + \alpha_1 r + \alpha_2 r^2 + \alpha_3 r^3 + \dots \quad (13)$$

and to insert this expression in the chiral equation for each model. Since the coefficients of each of the powers of  $r$  on the right-hand side of the equation must vanish individually, we find in the case of the Skyrme model (i.e. model  $\chi_S(a)$ ) that  $\alpha_n = 0$  for  $n = \text{even}$  and the following relations for the lowest  $n$ :

$$\alpha_3 = -\frac{\alpha_1^3}{30} \left[ \frac{4 + \alpha_1^2}{1 + \alpha_1^2} \right]$$

$$\alpha_5 = \frac{\alpha_1^5}{1400} \left[ \frac{40 + 32\alpha_1^2 + 11\alpha_1^4 + 7\alpha_1^6}{(1 + \alpha_1^2)^3} \right] \quad (14)$$

Thus all  $\alpha_i$  coefficients are determined once  $\alpha_1$  is fixed. We repeat this operation for  $r \rightarrow \infty$ , using:

$$F(r) = \frac{\beta_2}{r^2} + \frac{\beta_3}{r^3} + \frac{\beta_4}{r^4} + \dots + \frac{\beta_{10}}{r^{10}} \quad (15)$$

and find  $\beta_n = 0$  for  $n = \text{odd}$  and:

$$\beta_4 = 0, \quad \beta_6 = -\frac{\beta_2^3}{21}, \dots$$

The second step of the procedure is to adjust the parameters of  $c_i$  to these power series expansion using both limits:

$$F(r \rightarrow 0) = \pi \left[ 1 - \frac{(1 + c_3)}{\lambda} r + \frac{(\frac{1}{2} + \frac{3}{2}c_3 - c_5 - c_7)}{\lambda^3} r^3 - \dots \right]$$

$$F(r \rightarrow \infty) = \pi \left[ \frac{(\frac{1}{2} - c_3 - c_5)\lambda^2}{r^2} + \frac{(-\frac{3}{8} + \frac{3}{2}c_3 + \frac{5}{2}c_5 - c_7 - c_9)\lambda^4}{r^4} + \dots \right]$$

Equating the various powers of this expression with the  $\alpha_i$  and  $\beta_i$  coefficients of equations (13) and (15), we find:

$$c_3 = \frac{\alpha_1 \lambda}{\pi} - 1, \quad c_5 = -\frac{\beta_2}{\pi \lambda^2} + \frac{1}{2} - c_3, \quad c_7 = -\frac{\alpha_3 \lambda^3}{\pi} + \frac{1}{2} + \frac{3c_3}{2} - c_5, \dots$$

Thus all the free parameters of our function are fixed once we have made a choice for the scale  $\lambda$  and for the slopes  $\alpha_1$  and  $\beta_2$ . As we have pointed out, the value of those three parameters is determined by minimization of the soliton mass  $M_S$  (eq.(10)).

The calculations are performed up to order  $n = 10$  in (8). The results we have reached are listed in Table I, along with the margin of error with respect to numerical results whenever data was available. We notice that the agreement with numerical results varies within a wide range, depending on the model considered. The agreement is best when the order of the Lagrangian is low (namely for the Skyrme model and the first two Jackson et al. models), but for the other models the series does not seem to converge very fast. This observation is related to the appearance of a new feature in the solution as  $m$  increases. Numerical as well as analytical analysis show that the space derivative of the chiral angle  $F(r)$  begins to develop a cusp near  $r=1$ , exhibiting phase transition.

The fact that this feature appears in the  $r \sim \lambda$  region signals a limitation in our procedure. The overall shape of our analytic profile function is largely determined by its asymptotic behavior at  $r \rightarrow 0$  and  $r \rightarrow \infty$ , and we have no way of carrying a direct fit of the solution in-between. When the solution is smooth (i.e., when the order of the Lagrangian is low), this causes no difficulty; but when phase transition occurs we cannot expect strict agreement between numerical and analytical solutions to hold for all  $r$ .

Next, we consider the Atiyah-Manton approach applied to the  $N > 1$  spherically symmetric solution. Although they are not the lowest energy solutions (stable solitons), the approach leads to interesting results since time-ordering in eq. (3) is straightforward in these cases.

#### IV. THE N=2,3,... HEDGEHOG SOLUTIONS

Proceeding on to the B=2 solution of the Skyrme field equation, we consider the spherically symmetric case:

$$\rho = 1 + \frac{\lambda_1^2}{(t - T_1)^2 + r^2} + \frac{\lambda_2^2}{(t - T_2)^2 + r^2}$$

The minimal energy is obtained for  $\lambda_1 = \lambda_2 = \lambda$  and  $(T_1 - T_2)$  large but finite [8]. When  $(T_1 - T_2) \rightarrow \infty$ , the solution has a simple form:

$$F_{N=2}(r) = 2\pi \left( 1 + \frac{r}{\sqrt{r^2 + \lambda^2}} \right) = 2F_{N=1}(r) \quad (16)$$

and its energy turns to be larger only by a minute amount ( $\sim 0.5$  ppm). Since the introduction a parameter for a finite  $(T_1 - T_2)$  seems to bring very little improvement to the solution, we shall build our  $N > 1$  solutions on the solution with  $(T_1 - T_2) \rightarrow \infty$ .

Applying this function to the Skyrme model and minimizing the mass with respect to  $\lambda$ , we find  $M_S = 310.8 (F_\pi/e)$ , 1.2% higher than the exact result  $M_S = 307.1 (F_\pi/e)$ . Similarly, if we set  $F_{N=3}(r) = 3F_{N=1}(r)$ , we find the minimal mass to be  $M_S = 633.1 (F_\pi/e)$ , 3.0% higher than the exact value. We see that the results seem to decrease in accuracy as  $N$  increases.

We generalize our own approach to the  $N > 1$  problem by setting:

$$F_N(r, \lambda) = \sum_{n=1}^N F_1(r, \lambda_n)$$

We have considered various alternatives for  $\lambda_n$ . For instance:

$$F_N(r, \lambda) = \sum_{n=1}^N F_1(r, \lambda) = NF_1(r, \lambda)$$

$$F_N(r, \lambda) = \sum_{n=1}^N F_1(r, n\lambda) = F_1(r, \lambda) + \dots + F_1(r, N\lambda)$$

In the  $N = 2$  and  $N = 3$  cases, the lowest mass  $M_S$  is reached as  $\lambda_n = \lambda$  and  $\lambda_n = n\lambda$  respectively. We show the results in Table II for the  $N = 2$  case. Comparing the numerical

and analytical profiles we have obtained for  $F(r)$ , we have noticed a slight difference. While the latter remains smooth for all  $r$ , there is a sharper drop-off of the former near  $F(r) \cong \pi$ . This disagreement surely accounts for the poorer results we obtain (see Table II).

## V. ADVANTAGES AND LIMITATIONS OF THE APPROACH

The reliability of our approach seems to depend largely on the model considered. The conditions imposed on the  $c_i$  coefficients insure a good agreement with numerical results when  $r \rightarrow 0$  and  $r \rightarrow \infty$ . This agreement is conserved when the solution remains smooth for all values of  $r$ , as in the case of the  $N = 1$  soliton in the Skyrme and Jackson models (i.e., models in which the order of the Lagrangian is small). But discrepancies between analytical and numerical results begin to appear when we consider higher-order models and when the Skyrmon number  $N$  exceeds 1. Smaller scale structure then appear in the intermediate region ( $r \rightarrow \lambda$ ), and the Atiyah-Manton scheme does not have any special treatment of those cases.

In the multi-Skyrmion problem, the sharper drop-off of the profile function near  $F = N\pi$  is smoothed out. This may be caused by the fact that our function includes a single scale,  $\lambda$ . We have tried the alternative of using two different scales by setting  $F_{N=2}(r, \lambda) = F_{N=1}(r, \lambda) + F_{N=1}(r, 2\lambda)$ , but with little or no improvement in that respect.

The same type of arguments go for two-phase models. Although the phase transition of the solution is reproduced by our analytical function, this feature is not accurately fitted. This type of model actually seems to show chaotic behavior near  $r \sim \lambda$ . Numerical as well as analytical treatment show that when we solve the differential equation for the chiral angle, we find we must finely tune the boundary conditions for  $r \rightarrow 0$  for a solution to be reached. The reason is that any non-trivial solution must obey  $(r^{-2} \sin^2 F) = 1$  to a very good accuracy for the range of values  $0 < r < r_0 \cong 1$  almost independently from the rest of the differential equation, otherwise terms of the form  $(r^{-2} \sin^2 F)^m$  go to 0 or  $\infty$  for  $m$  large. Any account to parameterize the phase transition must take that into account, which is not the case in our procedure. We find that the  $\alpha_1$ ,  $\beta_2$  and  $\lambda$  parameters must also be finely tuned, and that the slopes at 0 and  $\infty$  do not always correspond to their numerical values.

Various ways of improving the fit of the solution may be taken into consideration. For instance, we have tried using other trial functions, not based on the Atiyah-Manton ansatz. The form of those is rather arbitrary however and, despite somewhat improved results, we have still met the same basic limitations. Another alternative we considered was to add a few more terms to our series in order to account for the coefficients  $\alpha_{11}$ ,  $\beta_{12}$ , etc. Unfortunately, besides the rapidly increasing complexity of our calculations, we have found that those terms contribute little to the  $r \sim \lambda$  region.

Since it seems impossible to carry out a direct fit of the solution in this intermediate region, another approach we are considering is to modify the corrective terms of the series, attempting to find a pattern in the values taken by the  $c_i$  coefficients that would make the series converge to a closed form.

Even though imperfect, we stress again that this approach is still probably the best alternative to complete numerical treatment, and that it can prove very useful whenever an analytical form of the solution is required, as we have already discussed with regard to stability analysis. The approach Igarashi et al. followed with the Skyrme model can easily



be generalized to other cases (two-phase structure models, multi-Skyrmion configurations, ...) by using a new family of trial functions based on our prototype.

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# TABLES

TABLE I. The values of the  $\alpha_1$  and  $\beta_2$  coefficients (see equations 13 and 15) and of some physical observables we have evaluated with our analytical ansatz in the  $N = 1$  case (single Skyrmion). The Skyrmion mass  $M_S$  (see equation 10) and the electrical radius  $\langle r^2 \rangle^{1/2}$  are expressed in units of  $\frac{F_\pi}{e}$  and  $\frac{1}{eF_\pi}$  respectively (with  $e$  and  $F_\pi$  defined in (2)), the pion decay constant in MeV and the axial coupling constant  $g_A$  is adimensional. We also give the margin of error with respect to exact numerical results as found in references [2-5].

$\chi(a)$	$\alpha_1$	$\beta_2$	$M_S$	$F_\pi$	$g_A$	$\langle r^2 \rangle^{1/2}$
$\chi_S(a)$	2.837 (0.10%)	1.075 (0.56%)	103.129 (0.0004%)	64.59 (0.03%)	-0.612 (0.10%)	0.7479 (0.04%)
$\chi_1(a)$	1.333	2.8062	136.505 (0.10%)	72.984 (0.02%)	-0.713243 (0.45%)	1.28496 (0.50%)
$\chi_{2,20}(a)$	0.981	2.050	73.5978 (3.59%)	99.29 (3.86%)	-0.743 (20.9%)	1.2090 (4.27%)
$\chi_{3,20}(a)$	0.970 (1.25%)	2.440	98.3687 (0.89%)	85.42 (0.75%)	-0.766 (8.27%)	1.2778 (1.47%)
$\chi_4(a)$	2.634	0.982	84.6796 (0.01%)	71.02	-0.697	0.7542 (0.03%)
$\chi_5(a)$	2.587	1.034	88.2485 (0.02%)	70.54	-0.703	0.7784 (0.24%)
$\chi_6(a)$	2.933	0.985	86.1798 (0.40%)	70.59	-0.697	0.7634 (0.07%)
$\chi_7(a)$	3.023	1.062	96.6275 (0.65%)	67.55	-0.663	0.7824 (0.18%)

TABLE II. The values of the  $\alpha_1$  and  $\beta_2$  coefficients (equation 13 and 15), of the Skyrmion mass  $M_S$  (equation 10) and of the observables  $F_\pi$ ,  $g_A$  and  $\langle r^2 \rangle^{1/2}$  as obtained with our analytical ansatz in the  $N = 2$  case. The units are the same as in Table I . The margin of error is given with respect to the numerical results we have obtained.

$\chi(a)$	$\alpha_1$	$\beta_2$	$M_S$	$F_\pi$	$g_A$	$\langle r^2 \rangle^{1/2}$
$\chi_S(a)$	5.800 (4.30%)	3.245 (0.37%)	308.569 (0.48%)	40.94	-0.525	1.472
$\chi_1(a)$	1.567	12.634	484.419	46.62	-0.555	2.993
$\chi_{2,20}(a)$	1.000	6.900	375.004	45.57	-0.717	2.5240
$\chi_{3,20}(a)$	0.957	9.400	447.664	43.98	-0.692	2.8388
$\chi_4(a)$	5.360	2.934	265.890	42.89	-0.535	1.4205
$\chi_5(a)$	5.400	3.070	273.490	42.78	-0.533	1.4515
$\chi_6(a)$	5.843	3.052	273.665	42.93	-0.546	1.5209
$\chi_7(a)$	6.111	3.063	294.382	41.72	-0.548	1.5346